MATHEMATICAL ECONOMICS PRACTICE PROBLEMS AND SOLUTIONS

Second Edition

G. Stolyarov II,
ASA, ACAS, MAAA, CPCU, ARe, ARC, API, AIS, AIE, AIAF

First Edition Published in March-April 2008
Second Edition Published in July 2014

Note: Here, I will present solve problems typical of those offered in a mathematical economics or advanced microeconomics course. The problems were originally compiled by Dr. Charles N. Steele and are reprinted with his generous permission. The solutions to the problems are my own work and not necessarily the only way to solve the problems.

Table of Contents

Section 1

Profit Maximization in Mathematical Economics

Problem 1. Suppose a firm faces a demand curve for its product P = a - bQ, and the firm's costs of production and marketing are C(Q) = cQ + d, where P is price, Q is quantity, and a, b, c, and d are positive constants. Find the following:

a. The formula for profit Π in terms of Q.

b. The first order condition (FOC) for maximum profit.

c. The second order condition (SOC) for maximum profit.

Solution 1a. $\Pi = TR - TC = PQ - C(Q) = aQ - bQ^2 - cQ - d =$ $\mathbf{\Pi = - bQ^2 + (a\text{-}c)Q - d}$

Solution 1b. FOC: $d\Pi/dQ = -2bQ + (a-c) \equiv 0$. Thus, $-2bQ = -(a-c)$ and $\mathbf{Q = (a\text{-}c)/2b}$.

Solution 1c. SOC: $\mathbf{d^2\Pi/dQ^2 = -2b < 0}$, since it is given that $b > 0$. Thus, $Q = (a-c)/2b$ is a maximum.

Problem 2. Suppose the firm faces a demand curve for its product P = 32 - 2Q, and the firm's costs of production and marketing are $C(Q) = 2Q^2$. Find the following.

a. The formula for profit Π in terms of Q.

b. The FOC and SOC for maximum total revenue.

c. The price and quantity that maximize total revenue, and the corresponding value of total revenue.

d. The FOC and SOC for maximum profit.

e. The price and quantity that maximize profit, and the corresponding value of profit.

f. What would the competitive price and quantity be, assuming $C(Q) = 2Q^2$ represented the industry cost function?

Solution 2a. $\Pi = TR - TC = PQ - C(Q) = 32Q - 2Q^2 - 2Q^2 =$ $\mathbf{\Pi = 32Q - 4Q^2}$

Solution 2b. $TR = 32Q - 2Q^2$

FOC: d[TR]/dQ = 32 - 4Q ≡ 0. Thus, **Q = 8**.

SOC: $\mathbf{d^2[TR]/dQ^2 = -4 < 0}$. Thus, Q = 8 is a maximum.

Solution 2c. The quantity that maximizes total revenue is **Q = 8,** according to the first and second-order conditions in Solution 2b. The price that maximizes total revenue is

32 - 2*8 = **P = 16.** Total revenue at this level is PQ = 16*8 = **TR =128.** We note that AVC here is 2Q = 2*8 = 16, so price is at least equal to average variable cost.

Solution 2d. FOC: dΠ/dQ = 32 - 8Q = 0. Thus, **Q = 4.**

SOC:$\mathbf{d^2\Pi/dQ^2 = -8 < 0}$. Thus, Q = 4 is a maximum.

Solution 2e. The quantity that maximizes profit is **Q = 4,** according to the first and second-order conditions in Solution 2d. The price that maximizes profit is

32 - 2*4 = **P = 24.** Total profit at this level is $32*4 - 4*4^2$ = **Π = 64**.

Here, 24 > 16, so P > AVC, and it is optimal for the firm to produce Q = 4.

Solution 2f. The firm will produce at P = MC, where P = 32 - 2Q. TC = $2Q^2$, so MC = 4Q. Thus, 32 - 2Q = 4Q. Thus, 32 = 6Q and Q = 32/6 = **Q = 16/3.** P = 32 - 2(16/3) = **P = 64/3**

Section 2

The Lagrangian Method of Constrained Optimization

Note: Here, I will present solve problems typical of those offered in a mathematical economics or advanced microeconomics course. The problems were authored by Dr. Charles N. Steele and are reprinted with his generous permission. The solutions to the problems are my own work and not necessarily the only way to solve the problems.

3. Find the maximum values of the objective function F subject to the given constraint for each of the following, using the Lagrangian method.

a. $F(x, y) = xy$, subject to $5x + 2y = 20$

b. $F(x, y) = 2x^{1/2}y^{1/2}$ subject to $x^2 + y^2 = 8$

c. $F(x, y, z) = xyz$ subject to $x^2 + y^2 + z^2 = 12$

d. $F(x, y, z) = x + y + z$ subject to $x^2 + y^2 + z^2 = 12$

Solution 3a. Lagrangian: $L(x, y, \lambda) = xy + \lambda[20 - 5x - 2y]$

$L_x = y - 5\lambda \equiv 0$

$L_y = x - 2\lambda \equiv 0$

$L_\lambda = 20 - 5x - 2y \equiv 0$

Thus, $2\lambda = x$ and $5\lambda = y$ (from the transformed for L_x and L_y).

So $20 - 5x - 2y = 20 - 5*2\lambda - 2*5\lambda = 20 - 20\lambda = 0$, so $20 = 20\lambda$ and $\boldsymbol{\lambda} =$ **1,**

whereby **$x = 2$ and $y = 5$**.

Solution 3b. Lagrangian: $L(x, y, \lambda) = 2x^{1/2}y^{1/2} + \lambda[8 - x^2 - y^2]$

$L_x = x^{-1/2}y^{1/2} - 2\lambda x \equiv 0$

$L_y = x^{1/2}y^{-1/2} - 2\lambda y \equiv 0$

$L_\lambda = 8 - x^2 - y^2 \equiv 0$

$x^{-1/2}y^{1/2} - 2\lambda x \equiv 0$ implies $2\lambda x = x^{-1/2}y^{1/2}$ and $2\lambda = x^{-3/2}y^{1/2}$

Thus, $\lambda = (1/2)x^{-3/2}y^{1/2}$

$x^{1/2}y^{-1/2} - 2\lambda y \equiv 0$ implies $2\lambda y = x^{1/2}y^{-1/2}$ and $2\lambda = x^{1/2}y^{-3/2}$

$x^{1/2}y^{-3/2} = x^{-3/2}y^{1/2}$ implies that $x^2 = y^2$ and thus $8 = 2x^2$ and $\mathbf{x = 2, y = 2}$.

$\lambda = (1/2)x^{-3/2}y^{1/2} = (1/2)(2)^{-3/2}(2)^{1/2} = \boldsymbol{\lambda = 1/4}$

Solution 3c. Lagrangian: $L(x, y, z, \lambda) = xyz + \lambda[12 - x^2 - y^2 - z^2]$

$L_x = yz - 2\lambda x \equiv 0$

$L_y = xz - 2\lambda y \equiv 0$

$L_z = xy - 2\lambda z \equiv 0$

$L_\lambda = 12 - x^2 - y^2 - z^2 \equiv 0$

$yz - 2\lambda x \equiv 0$ implies $2\lambda x = yz$ and $\lambda = yz/2x$

$xz - 2\lambda y \equiv 0$ implies $2\lambda y = xz$ and $\lambda = xz/2y$

$xy - 2\lambda z \equiv 0$ implies $2\lambda z = xy$ and $\lambda = xy/2z$

$yz/2x = xz/2y = xy/2z$ implies

$y^2z/x = xz = xy^2/z$ implies

$y^2z^2 = x^2z^2 = x^2y^2$

$x^2z^2 = x^2y^2$ implies $z^2 = y^2$

$y^2z^2 = x^2z^2$ implies $x^2 = y^2$

Thus, $x^2 + y^2 + z^2 = 12$ implies $3x^2 = 12$ and $\mathbf{x = 2, y = 2, z = 2}$

$\lambda = xy/2z = (2*2)/(2*2) = \boldsymbol{\lambda = 1}$

Solution 3d. Lagrangian: $x + y + z + \lambda[12 - x^2 - y^2 - z^2]$

$L_x = 1 - 2\lambda x \equiv 0$

$L_y = 1 - 2\lambda y \equiv 0$

$L_z = 1 - 2\lambda z \equiv 0$

$L_\lambda = 12 - x^2 - y^2 - z^2 \equiv 0$

Rearranging the expressions for L_x, L_y, and L_z, we get $x = y = z = 1/2\lambda$

Thus, $x^2 + y^2 + z^2 = 12$ implies $3x^2 = 12$ and **$x = 2, y = 2, z = 2$**

$z = 1/2\lambda$ means that $2\lambda = 1/z$ and $\lambda = 1/2z =$ **$\lambda = 1/4$**

Section 3

Intertemporal Allocation of a Depletable Resource: Optimization Using the Kuhn-Tucker Conditions

9. Intertemporal allocation of a depletable resource: suppose the demand for a resource in year t is $P_t = a - bq_t$, where t indexes the year, P is price, q is quantity demanded, and a and b are positive constants. Suppose also that the total cost of producing the resource in year t is $TC(q_t) = cq_t$, where c is a positive constant. Suppose also that the total amount of the resource is Q. The real interest rate is r. Assume that the objective is to maximize present value of net benefits to consumers consuming this resource (i.e., not a monopoly problem).

a. Set up the Lagrangian for this problem, assuming t = (0, 1, 2, ..., T)

b. Show what the Kuhn-Tucker FOC are for this problem.

c. Why are the Kuhn-Tucker conditions relevant, rather than equality constraints?

d. Solve this problem for a = 8, b = 0.4, c = 2, Q = 20, r = 0.05, and T = 2 (i.e., three period model.

e. The problem in (d) is a dynamic problem. What is meant by "dynamic?" How much would be consumed each period if Q = 100? What would be the value of relaxing the constraint Q = 20 in each period?

Solution 9a. For a time period t, net benefit to consumers can be expressed as

Total benefits - $TC(q_t) = {}_0^{qt}\int(a - bq)dq - cq_t = aq_t - (b/2)q_t^2 - cq_t$

Lagrangian: $\mathbf{L = {}_{t=0}^{T}\Sigma[(aq_t - (b/2)q_t^2 - cq_t)[1/(1+r)^t]] + \lambda[Q - {}_{t=0}^{T}\Sigma q_t]}$

Solution 9b. FOC:
$L_{q0} = (a - bq_0 - c) - \lambda \equiv 0$
$L_{q1} = (a - bq_1 - c)(1/(1+r)) - \lambda \equiv 0$
$L_{q2} = (a - bq_2 - c)(1/(1+r)^2) - \lambda \equiv 0$ [...]
$L_{qT} = (a - bq_T - c)(1/(1+r)^T) - \lambda \equiv 0$
$L_\lambda = Q - {}_{t=0}^{T}\Sigma q_t \geq 0, \lambda \geq 0.$

Solution 9c. The Kuhn-Tucker conditions are relevant, rather than the equality constraints, because it is possible to not consume the entire available stock of resources over the time period in question (in which case $Q - {}_{t=0}^{T}\Sigma q_t > 0$). Doing so may be optimal if individuals discount the future at a sufficiently low rate that the present value of the net benefits of this resource stock to them will be increased by deferring consumption.

Solution 9d. For a = 8, b = 0.4, c = 2, Q = 20, r = 0.05, and T = 2, we have the following FOC:

$L_{q0} = (8 - 0.4q_0 - 2) - \lambda \equiv 0$
$L_{q1} = (8 - 0.4q_1 - 2)(1/1.05) - \lambda \equiv 0$
$L_{q2} = (8 - 0.4q_2 - 2)(1/1.1025) - \lambda \equiv 0$
$L_\lambda = 20 - q_0 - q_1 - q_2 \geq 0, \lambda \geq 0.$

So $\lambda = 6 - 0.4q_0 = (1/1.05)(6 - q_1) = (1/1.1025)(6 - q_2)$

$q_0 = 6/0.4 - \lambda/0.4$

$1.05\lambda = 6 - q_1$

So $q_1 = 6/0.4 - 1.05\lambda/0.4$

$q_2 = 6/0.4 - 1.1025\lambda/0.4$

If $20 - q_0 - q_1 - q_2 = 0$, then $20 - 3(6/0.4) + (1 + 1.05 + 1.1025)\lambda/0.4 = 0$

Thus, $-25 + 7.88125\lambda = 0$ and $7.88125\lambda = 25$, which means that $\boldsymbol{\lambda = 3.172085646}$

$q_0 = 6/0.4 - 3.172085646/0.4 =$ $\boldsymbol{q_0 = 7.069785884}$

$q_1 = 6/0.4 - 1.05*3.172085646/0.4 =$ $\boldsymbol{q_1 = 6.673275178}$

$q_2 = 6/0.4 - 1.1025\lambda/0.4 =$ $\boldsymbol{q_2 = 6.256938937}$

Solution 9e. In this problem, the available stock of resources Q will determine whether $\lambda \geq 0$ or $\lambda = 0$. That is, one of the initial conditions will have an effect on what constraints apply to the problem. In the case where $\lambda \geq 0$, the stock of resources Q is a binding constraint, and consumption in one period will affect how much can be consumed in the subsequent periods. This makes the problem dynamic, because it is impossible to consume 15 units of the resource (the number that would be consumed if Q were not binding) each time period. What is consumed now necessarily limits what can be consumed in future periods.

If Q = 100, then the resource stock will not be exhausted within three time periods, because this would require on average $q_t = 100/3$, and $8 - 0.4*100/3 = -5.3333$, which is a negative price and thus impossible. Consumers simply do not demand this resource enough to be willing to consume 100 units of it over three periods. Thus, $100 - q_0 - q_1 - q_2 \geq 0$ and $\lambda = 0$. Thus, $6 = 0.4q_0$ and $\boldsymbol{q_0 = q_1 = q_2 = 15}$. (With $\lambda = 0$, the quantities consumed each time period will be equal.)

The value of relaxing the constraint Q = 20 in each period would be $\boldsymbol{\lambda = 3.172085646}$, as solved within the problem where Q = 20.

Section 4

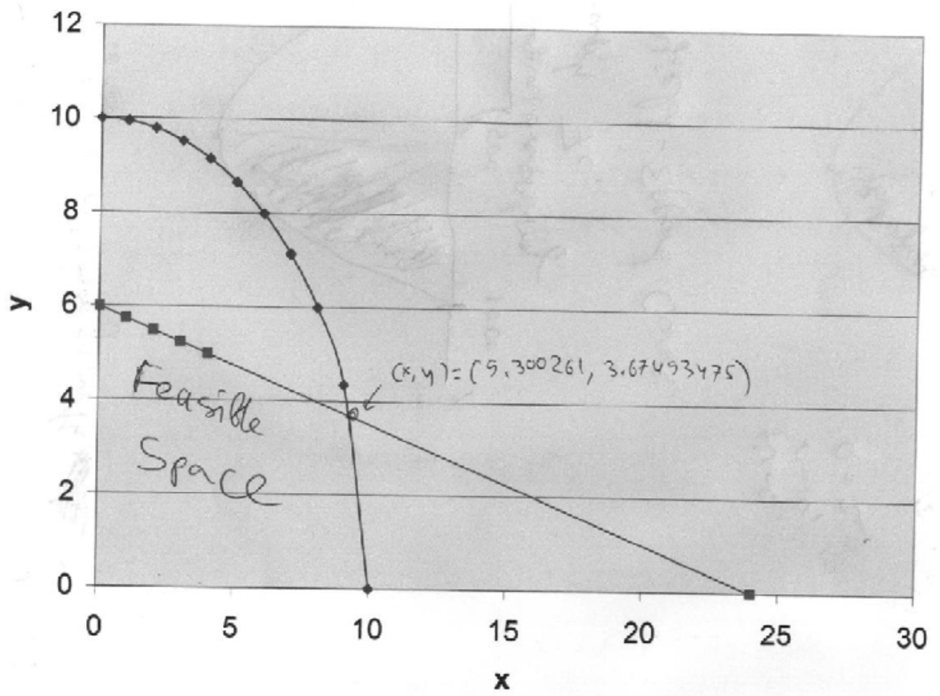
Optimization with Inequality Constraints

Problem 4. Use the Kuhn-Tucker conditions to find maxima for the following:
a. F(x, y) = xy, subject to (i) $x^2 + y^2 \leq 100$ and (ii) $x + 4y \leq 24$

b. F(x, y) = a*ln(x) + b*ln(y), subject to (iii) $x + y \leq 100$ and (iv) $x + 2y \leq 140$.

Assume that a + b = 1

Solution 4a. See the graph of the feasible space.



Note: The distorted quarter-circle on the graph is constraint (i); the diagonal line is constraint (ii).

The intersection of $x^2 + y^2 = 100$ and x + 4y = 24 where 4y = 24 -x, and y = 6 - x/4. Thus, $x^2 + (6 - x/4)^2 = 100$ and x = 9.300261, so y = 6 - 9.300261/4 = y = 3.67493475. At (x, y) = (9.300261, 3.67493475), xy = 34.17785233. We cannot, however, move to the right of (x, y) = (9.300261, 3.67493475) along the constraint $x^2 + y^2 = 100$ to find a higher value of xy, because the point

(x, y) = (9.300261, 3.67493475) is already well below the 45-degree line, at which the value of cos(θ)sin(θ) for *any* circle is maximized. Moving farther away from the 45-degree line by moving to the right will only reduce the value of 10cos(θ)sin(θ), which is equivalent to the value of xy along the quarter-circle denoted by $x^2 + y^2 = 100$.

Thus, we search for our maximum along the line x + 4y = 24 to the left of (x, y) = (9.300261, 3.67493475). This constraint can also be expressed as y = 6 - x/4.

But when we move to the left by Δ, we also increase y by Δ/4.

This means that xy becomes $(x - \Delta)(y + \Delta/4) = xy - \Delta y + x\Delta/4 - (\Delta)^2/4$

At y > 3.67493475, Δy > 3.67493475Δ

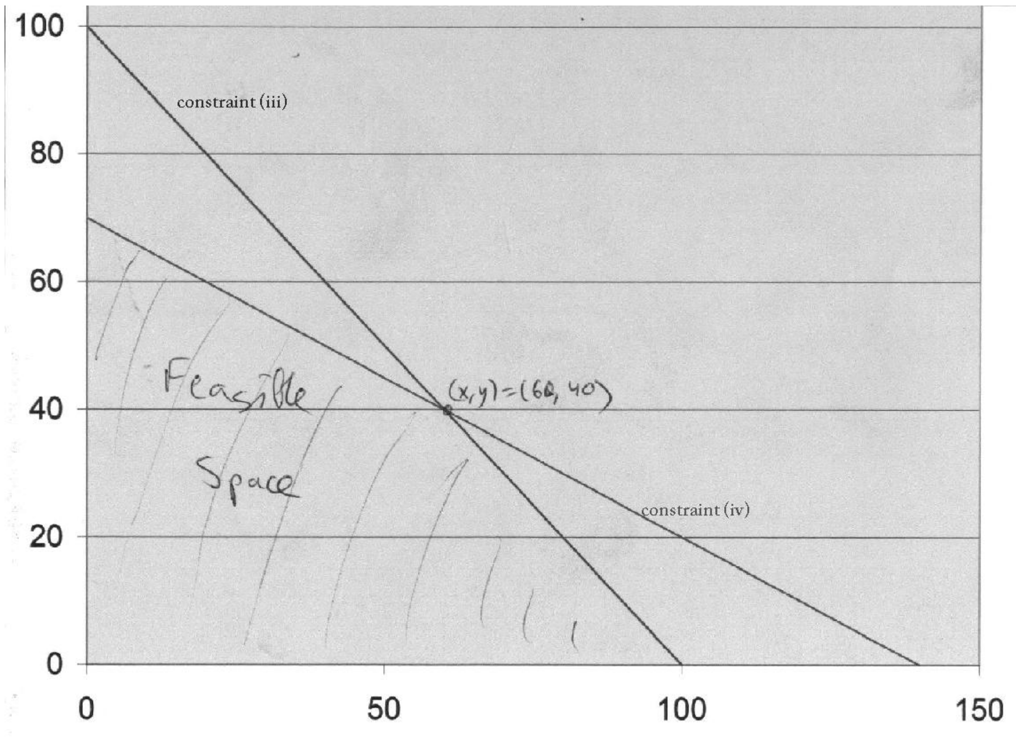
At x < 9.300261, Δx/4 < 2.32506525Δ < 3.67493475Δ

Thus, the positive xΔ/4 term will necessarily be smaller than the negative Δy term.

And so $xy - \Delta y + x\Delta/4 - (\Delta)^2/4 < 0$ for all Δ when we move to the left of (x, y) = (9.300261, 3.67493475). And moving inward into the feasible space from this point will simply diminish either x or y without increasing the other variable. Thus, since we cannot increase the value of xy by moving either to the left or to the right of (x, y) = (9.300261, 3.67493475), the maximum is

(x, y) = (9.300261, 3.67493475) and xy = 34.17785233.

Solution 4b. See the graph of the feasible space.



We first find the intersection of x + y = 100 and x + 2y = 140, which occurs at x = 100 - y = 140 - 2y, which means that y = 40 and x = 60. At x = 60, y = 40, a*ln(x) + b*ln(y) = 4.094344562a + 3.688879454b

Lagrangian: L(x, y, λ, μ) = a*ln(x) + b*ln(y) + λ[100 - x - y] + μ[140 - x - 2y].

FOC: L_x = a/x - λ - μ ≡ 0

L_y = b/y - λ - 2μ ≡ 0

L_λ = 100 - x - y ≥ 0, λ ≥ 0 with complementary slackness.

L_μ = 140 - x - 2y ≥ 0, μ ≥ 0 with complementary slackness.

Case I: If λ = 0 and μ > 0, then μ = a/x and 2μ = b/y. Thus, x = a/μ and y = b/2μ.

Then, since μ > 0, 140 = x + 2y = (a + b)/μ and μ = 1/140. Then, since a + b = 1,

x = 140a and y = 70b.

Then we need to check that the other constraint (iii) is satisfied:

100 ≥ x + y = 140a + 70b = 140 - 70b. If 100 ≥ 140 - 70b, then 70b ≥ 40 and b ≥ 4/7. So Case I is internally consistent if **b ≥ 4/7**. In that case, the maximum will be at **(x, y) = (140a, 70b)**.

Case II. If λ > 0 and μ = 0, then λ = a/x = b/y and x = a/λ while y = b/λ.

Then, since λ > 0, 100 = x + y = (a + b)/λ and λ = 1/100. Then, since a + b = 1, x = 100a and y = 100b. Then we need to check that the other constraint (iv) is satisfied:

140 ≥ x + 2y = 100a + 200b = 100 + 100b Thus, 40 ≥ 100b and b ≤ 2/5. So Case II is internally consistent if **b ≤ 2/5**. In that case, the maximum will be at **(x, y) = (100a, 100b)**

Case III. If λ > 0 and μ > 0, we get a maximum at **(x, y) = (60, 40).** This case is internally consistent if **2/5 < b < 4/7**.

Problem 5. What would be the value of relaxing each of the constraints (i), (ii), (iii), and (iv)?

Solution 5. The value of relaxing constraint (i) is λ in Problem 4a.

We take the Lagrangian for 4a:

L(x, y, λ, μ) = xy + λ[100- x^2 - y^2] + μ[24 - x - 4y]

FOC:

$L_x = y - 2x\lambda - \mu \equiv 0$

$L_y = x - 2y\lambda - 4\mu \equiv 0$

We know that at the maximum, (x, y) = (9.300261, 3.67493475),

so 3.67493475 = 18.600522λ + μ

and 9.300261 = 7.3498695λ + 4μ

Solving this system of equations by row-reducing a 2 x 3 matrix, we get,

λ = 0.0805264621 and **μ = 2.177100501**

The value of relaxing constraint (ii) is μ in Problem 4a. From the work above, **μ = 2.177100501.**

The value of relaxing constraint (iii) is λ in Problem 4b. If λ > 0 and this constraint is binding, then λ = a/x - μ.

If x = 60, y = 40, then λ + μ = a/60 and λ + 2μ = b/40, so μ = b/40 - a/60 and λ = a/60 - b/40 + a/60 = **λ = a/30 + b/40**.

If (x, y) = (100a, 100b), then

λ + μ = 1/100 and λ + 2μ = 1/100, so μ = 0 and **λ = 1/100**.

The value of relaxing constraint (iv) is μ in Problem 4b, which is either **μ = b/40 - a/60** (when x = 60, y = 40) or **μ = 1/140,** when (x, y) = (140a, 70b).

Section 5

The Economics of Fisheries

Problem 6. Logistic population growth: suppose the growth rate of a population of wild fish is given by $F(x) = 10x - 0.01x^2$ if the population is left undisturbed, were x is population.

a. What is the equilibrium natural population?
b. What is the maximum sustained yield, and the corresponding population?
c. Graph the population growth function. Which portions of this function correspond to biological overfishing?

Solution 6a. The equilibrium population level occurs when the growth rate of population is zero and the population is positive. That is, $10x = 0.01x^2$ and $10 = 0.01x$. Thus, **x = 1000 fish**

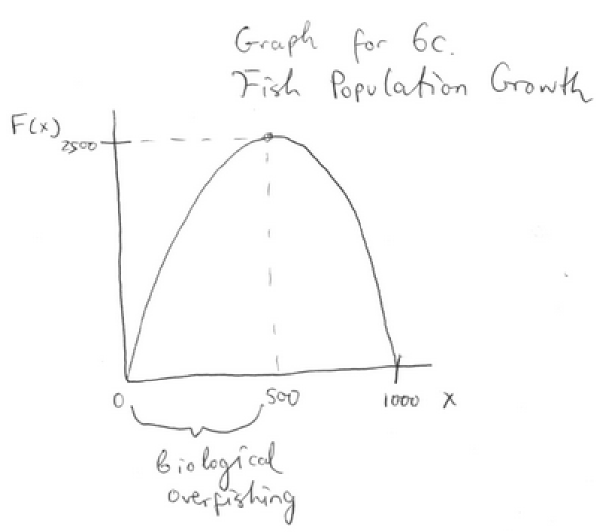
Solution 6b. The maximum sustained yield occurs at the maximum growth rate.

FOC: $dF/dx = 10 - 0.02x \equiv 0$, which means that **x = 500 at MSY**.

SOC: $F'' = -0.02 < 0$, so x = 500 is a maximum. The growth rate at MSY =

$10*500 - 0.01*500^2$ = **F(500) = 2500 fish/unit of time**. It will be possible to capture 2500 fish per unit of time while keeping the fishery population at 500 and then to capture another 2500 per unit of time indefinitely. So **MSY= 2500 fish.**

Solution 6c. See the graph below.



Biological overfishing corresponds to all populations less than (left of) x = 500.

Problem 7. Schaefer model: suppose that the yield (harvest h) is given by h = 2Ex ,where E is the amount of fishing effort and x is population (here q, the productivity parameter, equals 2).

a. Find sustainable population x as a function of effort E.

b. Find sustainable yield as a function of E.

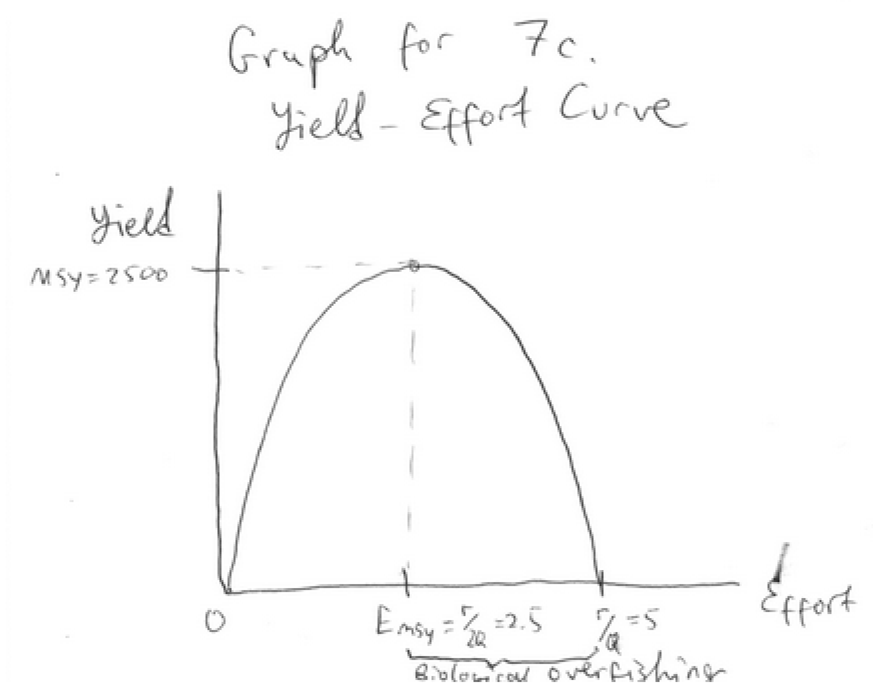
c. Graph the resulting Yield-Effort curve. Which portions of this curve correspond to biological overfishing?

Solution 7a. Now the net growth rate of fish is F(x) - h = 10x - $0.01x^2$ - 2Ex. The equilibrium population level occurs when the net growth rate of population is zero and the population is positive. That is, (10 - 2E)x = $0.01x^2$ and 10 - 2E = 0.01x. Thus,

x = 1000 - 200E

Solution 7b. The yield-effort curve is given by Y = qKE(1 - qE/r), where r =10, q = 2, K = 1000. Thus, Y = 2000E(1 - E/5) = **Y = 2000E - 400E^2**.

Solution 7c. See the graph below.



Biological overfishing corresponds to all levels of effort to the right of E_{MSY}.

8. Gordon model: Suppose price of fish P = 10, and total cost of fishing effort is TC(E) = 1000E.

a. Find the Total Revenue Product Curve.

b. Suppose the fishery is managed by an owner with exclusive property rights. Find the level of effort E that maximizes the rent from the fishery, and calculate the rent.

c. Find the level of effort that would occur under open access.

d. Under open access, is this fishery suffering from economic overfishing?
e. Under open access, is this fishery suffering from biological overfishing? Why or why not?

Solution 8a. TRP = P*Y(E). Y(E) = $2000E - 400E^2$ from 7b. So

TRP = $\mathbf{20000E - 4000E^2}$

Solution 8b. The rent-maximizing level of effort will be the level of effort that maximizes Rent = TRP - TC(E) = $20000E - 4000E^2 - 1000E = 19000E - 4000E^2$.

d[Rent]/dE = 19000 - 8000E ≡ 0 and **E = 19/8 = 2.375**.

Rent will be $19000*2.375 - 4000*2.375^2$ = **Rent = $22,562.50**

Solution 8c. Under open access, economic rent from the fishery would diminish to zero and $19000E - 4000E^2 = 0$, so $19000E = 4000E^2$ and the non-zero level of effort for this is

E = 19000/4000 = **E = 19/4 = 4.75**

Solution 8d. Under open access, the fishery **issuffering from economic overfishing,** because rents are being dissipated and the fishery is contributing no value on net to the economy. With less effort, rents could exist and the economic value of the fishery could be increased.

Solution 8e. Under open access, the fishery **is suffering from biological overfishing.** Biological overfishing occurs when E > r/2q. But here, r/2q = 2.5 = E, and 4.75 > 2.5, so the fishery is being fished at a level of effort greater than the one which corresponds to maximum sustainable yield.

Section 6

Additional Practice Problems Involving the Kuhn-Tucker Conditions

Note: You can use this equation solver to solve any single-variable equations after you set them up. The focus of these problems is not on algebraic manipulations, but on the concepts and the procedure involved.

Problem KTC1. Use the Kuhn-Tucker conditions to find maximum for the following:

$F(x, y) = 2x^2y^2$, subject to constraints (v) $18x + 3y \leq 1266$ and (vi) $2x + 5y \leq 1000$.

Assume that x and y must each be greater than or equal to 0.

Solution KTC1.

Lagrangian: $L(x, y, \lambda, \mu) = 2x^2y^2 + \lambda[1266 - 18x - 3y] + \mu[1000 - 2x - 5y]$

FOC: $L_x = 4xy^2 - 18\lambda - 2\mu \equiv 0$. ($x*L_x = 0$, but if x is zero, then $F(x, y) = 0$, so L_x must be 0).

$L_y = 4x^2y - 3\lambda - 5\mu \equiv 0$. ($y*L_y = 0$, but if y is zero, then $F(x, y) = 0$, so L_y must be 0).

$L_\lambda = 1266 - 18x - 3y \geq 0$, $\lambda \geq 0$ with complementary slackness.

$L_\mu = 1000 - 2x - 5y \geq 0$, $\mu \geq 0$ with complementary slackness.

Case I: We first find the intersection of $18x + 3y = 1266$ and
$2x + 5y = 1000$, which occurs when $3y = 1266 - 18x$ and $y = 422 - 6x$. Thus,
$2x + 5(422 - 6x) = 1000$ and $x = 555/14 = x = 39.64285714$, so $y = 422 - 6x =$
$y = 184.1428571$. Here, $F(39.64285714, 184.1428571) = 106{,}578{,}510.2$.

Case II: If $\lambda = 0$ and $\mu > 0$, then
$4x^2y - 5\mu = 0$ and $4xy^2 - 2\mu = 0$, so $4x^2y = 5\mu$ and $4xy^2 = 2\mu$, so $(y/x) = 2/5$ and so $5y = 2x$ Since $\mu > 0$, $L_\mu = 0$, so $1000 - 2x - 5y = 0 = 1000 - 4x$, so $4x = 1000$ and $x = 250$, while $y = (2/5)x = y = 100$. But $(x, y) = (250, 100)$ is outside of our feasible space, since it violates the constraint $18x + 3y \leq 1266$, as $18*250 + 3*100 = 4800 > 1266$.

Case III: If $\mu = 0$ and $\lambda > 0$, then $4xy^2 - 18\lambda = 0$ and $4xy^2 = 18\lambda$.
Likewise, $4x^2y - 3\lambda = 0$, so $4x^2y = 3\lambda$. Thus, $(y/x) = 6$ and $y = 6x$, implying that $3y = 18x$.
Since $\lambda > 0$, $L_\lambda = 0$, so $1266 - 18x - 3y = 0$ and $1266 - 36x = 0$, whereby $x = 35.166666667$
And $y = 6x = 211$. This is within the constraint $18x + 3y \leq 1266$, but violates the constraint

$2x + 5y \leq 1000$, as 2*35.166666667 + 5*211 = 1125.3333333 > 1000. Thus, this solution is outside our feasible space. Hence, our function-maximizing values are **(x, y) = (39.64285714, 184.1428571)**, whereby **F(x, y) = 106,578,510.2.**

Problem KTC2. Use the Kuhn-Tucker conditions to find maximum for the following:

$F(x, y) = xy$, subject to constraints (vii) $x^2 + y^2 \leq 3481$ and (viii) $2x + 4y \leq 120$.

Assume that a and b are constants and that x and y must each be greater than or equal to 0.

Solution KTC2.

Lagrangian: $L(x, y, \lambda, \mu) = ax^3y + bxy^3 + \lambda[3481 - x^2 - y^2] + \mu[120 - 2x - 4y]$.

FOC: $L_x = y - 2\lambda x - 2\mu \equiv 0$

($x*L_x = 0$, but if x is zero, then F(x, y) = 0, so L_x must be 0).

$L_y = x - 2\lambda y - 4\mu \equiv 0$

($y*L_y = 0$, but if y is zero, then F(x, y) = 0, so L_y must be 0).

$L_\lambda = 3481 - x^2 - y^2 \geq 0$, $\lambda \geq 0$ with complementary slackness.

$L_\mu = 120 - 2x - 4y \geq 0$, $\mu \geq 0$ with complementary slackness.

Case I: We first find the intersection of $x^2 + y^2 = 3481$ and $2x + 4y = 120$.
$4y = 120 - 2x$, so $y = 30 - 0.5x$ and $x^2 + (30 - 0.5x)^2 = 3481$. This gives a positive solution for x as x = 58.9979, implying that y = 30 - 0.5*58.9979 = y = 0.50105. When (x, y) = (58.9979, 0.50105), F(x, y) = 29.5608978

Case II: If $\lambda = 0$ and $\mu > 0$, then $y - 2\mu \equiv 0$ and $2\mu = y$
Likewise, $x - 4\mu \equiv 0$, so $4\mu = x$ and $x = 2y$. Since $\mu > 0$, $L_\mu = 0$, so $120 - 2x - 4y = 0$ and $120 - 4x = 0$, so x = 30 and y = 15. (x, y) = (30, 15) meets constraint (vii), as $30^2 + 15^2 \leq 3481$ and meets constraint (viii), as 2*30 + 4*15 = 120. F(x, y) = 30*15 = 450

Case III: If $\mu = 0$ and $\lambda > 0$, then $x - 2\lambda y = 0$ and $x = 2\lambda y$, so $2\lambda = x/y$.

Likewise, $y - 2\lambda x = 0$ and $2\lambda = y/x$. Thus, x/y = y/x, so x = y and $3481 - x^2 - y^2 = 0$

(Since $\lambda > 0$, $L_\lambda = 0$). Thus, $3481 = 2x^2$, and x = y = 41.71930009. But this fails to meet constraint (viii), since 2*41.71930009 + 4*41.71930009 = 250.3158005 > 120.

Thus, the maximum for F(x, y), given the inequality constraints, is **(x, y) = (30, 15)** and **F(x, y) = 450**.

Section 7

Additional Problems on the Economics of Fisheries

Problem EF1. The population of magenta-spotted dodecahedral fish in a particular fishery follows the growth function $F(x) = 987x - 0.45x^2$ if the population is left undisturbed, were x is population. What is the equilibrium natural population? Assume that fractional fish are possible.

Solution EF1. The equilibrium population level occurs when the growth rate of population is zero and the population is positive. That is, $0 = 987x - 0.45x^2$ and

$987x = 0.45x^2$, so x = 987/0.45 = **x = 2193.3333333333 fish**

Problem EF2. For the fishery in Problem EF1, what is the maximum sustained yield, and the corresponding population?

Solution EF2. We seek to maximize $F(x) = 987x - 0.45x^2$.

FOC: We take $F'(x) = 987 - 0.9x \equiv 0$, so 0.9x = 987 and x = 987/0.9 =

x_{MSY} = 1096.666667.

SOC: $F''(x) = - 0.9 < 0$, so x_{MSY} is indeed a maximum.

MSY = F(1096.666667) = 987*1096.666667 - 0.45*1096.666667^2 = **MSY = 541205 fish/unit of time**. It will be possible to capture 541205 fish per unit of time while keeping the fishery population at 1096.666667 and then to capture another 541205 per unit of time indefinitely.

Problem EF3. Humans discover the magenta-spotted dodecahedral fish fishery and begin to harvest at a yield of h = 8Ex, where E is the amount of fishing effort and x is population. Find sustainable population x as a function of effort E.

Solution EF3. The net rate of fish population growth is now

$F(x) - h = 987x - 0.45x^2 - 8Ex$. At the sustainable population, the net rate of fish population growth is zero. Thus, $(987 - 8E)x - 0.45x^2 = 0$ and x = (987 - 8E)/0.45 =

x = 2193.33333333 - 17.7777777777E

Problem EF4. Using the information from Problems EF1-3, find sustainable yield as a function of E.

Solution EF4. Y(E) = qEx = 8Ex in this case. From Solution EF3, the sustainable population x is 2193.33333333 - 17.7777777777E. Thus, sustainable yield is
8E(2193.33333333 - 17.7777777777E) = **Y(E) = 17546.666667E - 142.22222222E^2**.

Problem EF5. Within the market for magenta-spotted dodecahedral fish, the price of fish is P = $50 per unit, and the total cost of fishing effort is TC(E) = 25000E. Find the Total Revenue Product Curve, expressed as a function of E. Use any relevant information from Problems EF1-4.

Solution EF5. TRP = P*Y(E). Y(E) = 17546.666667E - 142.22222222E^2 from Solution EF4, so TRP = 50(17546.666667E - 142.22222222E^2) =

TRP = 877333.3333E - 7111.1111111E^2.

Problem EF6. Suppose that Multinational Corp. is the exclusive property owner of the magenta-spotted dodecahedral fish fishery and seeks to maximize its rents. Find the level of effort E that maximizes the rent from the fishery, and calculate the rent. Use any relevant information from Problems EF1-5.

Solution EF6. Rent = TRP - TC(E) = 877333.3333E - 7111.1111111E^2 - 25000E
Thus, Rent = 852333.3333E - 7111.1111111E^2 and
FOC: d[Rent]/dE = 852333.3333 - 14222.2222222E ≡ 0, so 14222.2222222E = 852333.3333, so
E = 59.9296875
SOC: Rent" = - 14222.2222222 < 0, so the E determined above is a maximum.

Problem EF7. Suppose that magenta-spotted dodecahedral fish "liberationists" take over the world and declare all fisheries to be open-access commons. Find the level of fishing effort that would occur under open access.

Solution EF7. Under open access, rents are dissipated to zero and thus
Rent = 852333.3333E - 7111.1111111E^2 = 0, so 852333.3333 = 7111.1111111E and
E = 119.859375

Problem EF8. Under open access, is the magenta-spotted dodecahedral fish fishery suffering from biological overfishing? Is it suffering from economic overfishing?

Solution EF8. The level of effort corresponding to maximum sustainable yield is the E that maximizes Y(E) = 17546.666667E - 142.22222222E^2.
Y'(E) = 17546.666667 - 284.44444444E ≡ 0, so E_{MSY} = 61.6875001**.** But
$E_{open\text{-}access}$ = 119.859375 > E_{MSY}, so **there exists biological overfishing.**

Under open access, the fishery **is suffering from economic overfishing,** because rents are being dissipated and the fishery is contributing no value on net to the economy. With less effort, rents could exist and the economic value of the fishery could be increased.

Section 8

The Deacon Model of Forest Economics

These problems use as their basis the economic model developed in Robert T. Deacon's paper, "The Simple Analytics of Forest Economics."

Problem ETH1. The trees in Forest Φ grow such that the volume of timber at time t can be represented as $f(t) = 900t - 50t^2 + 60t^3 + 7t^4$. (Note that these trees do not follow a typical biological growth function). The annual real interest rate is 0.19, and each unit of timber can be sold at a price of 100 Yap pieces of stone (YPS) net of harvest costs. Assume there is no opportunity cost to using the land to grow timber. At t = 89 years from the beginning of the trees' growth, Imhotep obtained sole ownership of the forest through a surprise bequest. What is the marginal benefit Imhotep would get by waiting to cut the trees for another year?

Solution ETH1. MB(waiting) = p*Δf(t), where p = 100 and Δf(t) = f(90) - f(89).

Thus, MB = 100(502686000 - 481177877) = MB = **2,150,812,300 YPS**

Problem ETH2. The trees in Forest Φ grow such that the volume of timber at time t can be represented as $f(t) = 900t - 50t^2 + 60t^3 + 7t^4$. (Note that these trees do not follow a typical biological growth function). The annual real interest rate is 0.19, and each unit of timber can be sold at a price of 100 Yap pieces of stone (YPS) net of harvest costs. Assume there is no opportunity cost to using the land to grow timber. At t = 89 years from the beginning of the trees' growth, Imhotep obtained sole ownership of the forest through a surprise bequest. What is the marginal cost he would incur by waiting to cut the trees for another year? Should he wait another year to cut down the trees?

Solution ETH2. MC(waiting) = r*p*f(t) = 0.19*100*f(89) = **MC = 9,142,379,663 YPS**

We note that 9,142,379,663 > 2,150,812,300, so MC > MB, and the owner would incur a much higher cost by waiting another year to cut down and sell the timber than he would gain in terms of marginal benefits. Thus, the owner **should not wait another year** to cut down the trees.

Problem ETH3. Now assume that, in a parallel universe to that in Problems ETH1-2, all other things are equal, but the interest rate is different such that t = 89 is the exact optimal time to harvest the trees. What is the annual interest rate r? Assume that Imhotep can only make a decision to harvest the trees once per year.

Solution ETH3. At the optimal harvest time, r = Δf(t)/f(t). Here, Δf(t) = f(90) - f(89), so

r = (f(90) - f(89))/f(89) = (502686000 - 481177877)/481177877 = **r = 0.0446989025**

Problem ETH4. The trees in Forest Ψ (another unusual forest - both biologically and in its nomenclature) grow such that the volume of timber at time t can be represented as $f(t) = e^t - 2^t$. The annual real interest rate is 0.07, and the per-year opportunity cost of using the forest land to grow timber is 36000 platinum hexagons (PH). Each unit of timber can be sold at a price of 900 PH. Tlaloc becomes the owner of the forest at t = 7 years. What is his marginal benefit of waiting another year to cut the trees?

Solution ETH4. MB(waiting) = p*Δf(t) = 900*(f(8) - f(7)) =

900(2724.95798704 - 968.633158429) = **MB = 1,580,692.346 PH.**

Problem ETH5. The trees in Forest Ψ (another unusual forest - both biologically and in its nomenclature) grow such that the volume of timber at time t can be represented as $f(t) = e^t - 2^t$. The annual real interest rate is 0.07, and the per-year opportunity cost of using the forest land to grow timber is 36000 platinum hexagons (PH). Each unit of timber can be sold at a price of 900 PH. Tlaloc becomes the owner of the forest at t = 7 years. What is his marginal cost of waiting another year to cut the trees? Should he wait another year to cut down the trees?

Solution ETH5. Here, the per-period cost of waiting is rpf(t) + R, where R = 36000. Thus, MC(waiting) = 0.07*900*f(7) + 36000 = 0.07*900*968.633158429 + 36000 =

MC = 97023.88898 PH. Here, MB > MC, so Tlaloc **should wait another year** to harvest the trees.

Section 9

The Second-Order Conditions for Multiple Choice Variables

Problem: Explain the second order conditions in the case 1, 2, and n > 2 choice variables for a maximum or minimum in an unconstrained choice problem. What is the relationship with concavity or convexity of the objective function?

Solution:

Second-order conditions for 1 choice variable, where z = F(x):

Maximum: $d^2z < 0$
Minimum: $d^2z > 0$

Second-order conditions for 2 choice variables, where z = F(x, y):

Maximum: $d^2z < 0$ iff $F_{xx} < 0$, $F_{yy} < 0$, and $F_{xx}F_{yy} > (F_{xy})^2$
Minimum: $d^2z > 0$ iff $F_{xx} > 0$, $F_{yy} > 0$, and $F_{xx}F_{yy} > (F_{xy})^2$

Second-order conditions for n choice variables, where z = F(x_1, x_2, ..., x_n):

Definition: A matrix is **positive definite** if all of its eigenvalues are greater than 0. Equivalently, a matrix is positive definite if all upper left corners of the matrix have positive determinants.

A matrix is **negative definite** if all of its eigenvalues are less than 0.

Let x_1, x_2, ..., x_n be the choice variables under consideration. And let all second partial derivatives of the function F exist with respect to the choice variables. Then the **Hessian** of F is the matrix of second partial derivatives defined as follows

$H(F)_{ij}(x) =$

[$F_{(x\_1)(x\_1)}$ $F_{(x\_1)(x\_2)}$ ... $F_{(x\_1)(x\_n)}$]

[$F_{(x\_2)x\_1)}$ $F_{(x\_2)(x\_2)}$ ... $F_{(x\_2)\,(x\_n)}$]

... ...

[$F_{(x\_n)\,(x\_1)}$ $F_{(x\_n)\,(x\_2)}$ ... $F_{(x\_n)\,(x\_n)}$], where $F_{(x\_i)\,(x\_j)}$ is the partial derivative of F with respect to x_i and x_j.

If at a point K, the Hessian of F is negative definite (i.e., $d^2z < 0$), then F has a local maximum at K.

If at a point K, the Hessian of F is positive definite (i.e., $d^2z > 0$), then F has a local minimum at K.

If the Hessian of F has both positive and negative eigenvalues at point K, then K is a saddle point for F.

Relationship between second order conditions and concavity or convexity of the objective function:

If the objective function is always concave, then any extreme point will be a maximum.

If the objective function is always convex, then any extreme point will be a minimum.

Section 10

Second-Order Conditions: Practice Problems and Solutions

For all problems here, find the extrema for each of the following functions, and identify whether the extrema are maxima or minima using second order conditions.

Problem 10-1.

$z = x^2 + xy + 2y^2 + 23$

Solution 10-1.
First-order conditions (FOC):
$z_x = 2x + y = 0$
$z_y = x + 4y = 0$

We can solve this system by row-reducing this matrix:

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 4 & 0 \end{array}\right]$$ with the following result:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$, which means that an extreme point is $(x^*, y^*) = (0, 0)$

Second-order conditions (SOC): $z_{xx} = 2$, $z_{xy} = 1$, $z_{yy} = 4$.

$z_{xx} > 0$, $z_{yy} > 0$, $z_{xx}z_{yy} = 8$, $(z_{xy})^2 = 1$, so $z_{xx}z_{yy} > (z_{xy})^2$ and **$(x^*, y^*) = (0, 0)$ is a minimum**.

Problem 10-2.
$z = -x^2 + xy - y^2 + 2x + y$

Solution 10-2.

FOC:
$z_x = -2x + y + 2 = 0$ and so $-2x + y = -2$
$z_y = x - 2y + 1 = 0$ and so $x - 2y = -1$

We can solve this system by row-reducing this matrix:

$$\left[\begin{array}{cc|c} -2 & 1 & -2 \\ 1 & -2 & -1 \end{array}\right]$$ with the following result:

$\left[\begin{array}{cc|c} 1 & 0 & 5/3 \\ 0 & 1 & 4/3 \end{array}\right]$, which means that an extreme point is (x*, y*) = (5/3, 4/3)

SOC: $z_{xx} = -2$, $z_{yy} = -2$, $z_{xy} = 1$.

$z_{xx} < 0$, $z_{yy} < 0$, $z_{xx}z_{yy} = 4$, $(z_{xy}) = 1$, so $z_{xx}z_{yy} > (z_{xy})$and **(x*, y*) = (5/3, 4/3) is a maximum**.

Problem 10-3.

$z = 8x^3 + 2xy - 3x^2 + y^2 + 1$

Solution 10-3.

FOC:
$z_x = 24x^2 + 2y - 6x = 0$
$z_y = 2x + 2y = 0$

$2x + 2y = 0$ implies that $y = -x$ and so $24x^2 + 2y - 6x = 0$ can be written as $24x^2 - 8x = 0$ and $8x(3x - 1) = 0$. Thus, x = 0 or x = 1/3. If x = 0, then y = -0 = 0. If x = 1/3, then y = -1/3. So the two possible extrema are

(x*, y*) = (0, 0) and (x*, y*) = (1/3, -1/3)

SOC: $z_{xx} = 48x - 6$, $z_{yy} = 2$, $z_{xy} = 2$.

If (x*, y*) = (0, 0), then $z_{xx} = -6 < 0$, but $z_{yy} = 2 > 0$, so (x*, y*) = (0, 0) is neither a minimum nor a maximum; rather, **(x*, y*) = (0, 0) is a saddle point**.

If (x*, y*) = (1/3, -1/3), then $z_{xx} = 10 > 0$, $z_{yy} = 2 > 0$, $z_{xy}^2 = 4$, and $z_{xx}z_{yy} = 20 > 4$, so $z_{xx}z_{yy} > z_{xy}^2$ and

(x*, y*) = (1/3, -1/3) is a minimum.

Problem 10-4. $z = e^{2x} - 2x + 2y^2 + 3$

Solution 10-4.
FOC:
$z_x = 2e^{2x} - 2 = 0$
$z_y = 4y = 0$
4y = 0 implies that y = 0.
$2e^{2x} - 2 = 0$ implies that $2e^{2x} = 2$ and $e^{2x} = 1$, so x = ln(1)/2 = 0. So an extreme point is (x*, y*) = (0, 0)

SOC: $z_{xx} = 4e^{2x}$, $z_{yy} = 4 > 0$, $z_{xy} = 0$. At (x*, y*) = (0, 0), $z_{xx} = 4 > 0$. Thus, $z_{xx}z_{yy} = 16 > z_{xy}^2 = 0$, so **(x*, y*) = (0, 0) is a minimum**.

Section 11

Expected Utility

Problem 11-1. Roger, a Von Neumann-Morgenstern utility maximizer, is planning a cross country trip. His utility from the trip is a function of how much of his cash Y he spends, and is given by

u(Y) = log Y (this is the base 10 logarithm) Assume Roger has $10000 to spend.

11-1a. What is his utility if he spends all his cash?

Solution 11-1a. Assume Roger has cash holdings equal to $Y. If he spends all his cash, Roger's utility will be log(Y) **=** log(10000) = **4**

11-1b. Suppose there is a 25% probability he will lose $1000 during the trip. What is his expected utility?

Solution 11-1b. Roger has a 0.25 probability of spending Y - 1000 cash and a 0.75 probability of spending Y cash. Thus, his expected utility is

0.75log(Y) + 0.25log(Y - 1000) = 0.75log(10000) + 0.25log(9000) = **3.988560627**

11-1c. Suppose Roger can buy insurance against this loss. What is the actuarially fair premium? Show that he will have higher utility with such insurance than without it.

Solution 11-1c. If Roger has a 0.25 probability of losing $1000, then Roger's expected loss is 0.25*1000 + 0.75*0 = $250. The actuarially fair premium should be equal to the expected loss, so the premium should be **$250.**

With this premium, Roger will have a guaranteed amount of cash to spend, equal to Y - 250, so his utility will be log(Y - 250) = log(10000 - 250) = log(9750) = **3.989004616.**

Since 3.989004616 > 3.988560627, Roger's utility will be greater with insurance than without.

11-1d. What is the maximum he would be willing to pay for this insurance?

Solution 11-1d. The maximum Roger would be willing to pay for this insurance will be the premium for which his expected utility will be 3.988560627 and his expected wealth will be $10^{3.988560627}$ = 9740.037464. Thus, Roger will be willing to pay at most 10000 - 9740.037464 = **$259.9625357**

11-1e. Suppose people with insurance behave more recklessly than those who do not, and therefore have a probability of 30% of losing $1000. What is the actuarially fair premium? Will Roger buy such insurance?

Solution 11-1e. With insurance, Roger's expected wealth becomes prior to the compensation becomes 0.7Y + 0.3(Y - 1000), and the expected loss is 0.3*1000 = $300, so the actuarially fair premium is **$300.**

To see whether Roger would buy such insurance, we compare the expected utility derived from the insurance: log(Y - 300) = log(9700) = 3.986771734 to the expected utility without insurance: 3.988560627

Since 3.986771734 < 3.988560627, **Roger will not buy the insurance.**

Problem 11-2. The Hillsdale city council desires to reduce the number of incidences of illegal parking. They are attempting to decide between two mutually exclusive programs. The first would raise parking fines by 10%. The second would increase enforcement, so that the probability a lawbreaker is caught rises by 10%. True or false: if lawbreakers are risk averse, the 10% increase in fines will have a greater disincentive effect than the 10% increase in enforcement rates.

Solution 11-2. Assume that the initial fine is F and the initial probability of a lawbreaker getting caught is P.

Plan 1: When fines are increased by 10%, F becomes 1.1F.

Plan 2: When probability of a lawbreaker getting caught increases by 10%,

P becomes 1.1P.

Assume that a lawbreaker has an expected utility function for his wealth of u(W) if he does not have to pay any fines and has no probability of getting caught. This function has a positive decreasing slope, since the lawbreaker is risk-averse.

Original expected utility EU_0 was

equal to P*u(W - F) + (1-P)*u(W)

Under Plan 1, expected utility is EU_1 = P*u(W - 1.1F) + (1-P)*u(W)

Under Plan 2, expected utility is EU_2 = (1.1P)*u(W - F) + (1-1.1P)*u(W)

EU_0 - EU_1 = [P*u(W - F) + (1-P)*u(W)] - [P*u(W - 1.1F) + (1-P)*u(W)] =

EU_0 - EU_1 = P*u(W - F) - P*u(W - 1.1F)

$EU_0 - EU_2 = [P*u(W - F) + (1-P)*u(W)] - [(1.1P)*u(W - F) + (1-1.1P)*u(W)] =$

$-0.1Pu(W - F) + 0.1Pu(W) = 0.1P[u(W) - u(W - F)]$

Now we must resolve the question of which is greater,

$P*u(W - F) - P*u(W - 1.1F)$, or $0.1P[u(W) - u(W - F)]$?

u(W) can be any function with decreasing positive slope. For convenience, we can assume that u(W) = ln(W). This can be done without loss of generality, since the ordinal ranking of $EU_0 - EU_1$ and $EU_0 - EU_2$ will be the same for any function with decreasing positive slope.

Then $P*u(W - F) - P*u(W - 1.1F) = P\ln(W - F) - P\ln(W - 1.1F) = P[\ln(W - F) - \ln(W - 1.1F)] = P[\ln[(W-F)/(W - 1.1F)] = P[\ln(1 + F/(W - 1.1F)]$

$0.1P[u(W) - u(W - F)] = 0.1P[\ln(W) - \ln(W - F)] = 0.1P[\ln[(1 + F/(W-F)]]$

We know that $W - 1.1F < W-F$, so $F/(W - 1.1F) > F/(W-F)$ and

$1 + F/(W - 1.1F) > 1 + F/(W-F)$. Thus,

$\ln[1 + F/(W - 1.1F)] > \ln[(1 + F/(W-F)]$

Of course, $P > 0.1P$, so

$P*\ln[1 + F/(W - 1.1F)] > 0.1P*\ln[(1 + F/(W-F)]$

and thus $P*u(W - F) - P*u(W - 1.1F) > 0.1P[u(W) - u(W - F)]$, so

$EU_0 - EU_1 > EU_0 - EU_2$ and the lawbreaker has a greater decrease in expected utility from Plan 1 than from Plan 2. Hence, when probability of a lawbreaker getting caught increases by 10%, the lawbreaker has a smaller disutility than he gets from a 10% increase in fines. Hence, the original statement is **true.**

Problem 11-3. A risk averse man faces two possible states of the world. In state 1, his income is Y_1 and in state 2 his income is Y_2. The states occur with probability p and 1 - p respectively. True or false: His indifference curves between income in state 1 and income in state 2 are convex to origin.

Solution 11-3. Let u(W) be a function with decreasing positive slope representing the utility of wealth W for the risk-averse man.

Expected utility $EU_1 = pu(Y_1) + (1-p)u(Y_2)$ for the situation where the risk-averse man faces uncertainty. If he were to get a guarantee of the expected value of this bet, then his expected utility would be $EU_2 = u[pY_1 +(1-p)(Y_2)]$

Since the man is risk-averse, for him $EU_2 > EU_1$, so

$u[pY_1 +(1-p)(Y_2)] > pu(Y_1) + (1-p)u(Y_2)$

This condition is the condition for *strict concavity* of a person's utility function.

However, we can graph this problem to show that this risk-averse man's indifference curves are convex.

Draw a set of axes with Y_1 on the horizontal axis and Y_2 on the vertical axis.

Label $Y_2 /(1-p)$ on the vertical axis and Y_1/p on the horizontal axis.

Draw a line between those two points.

Then is a kind of "budget constraint" with slope $-[Y_2 /(1-p)]/[Y_1/p] = -[p/(1-p)][Y_2/Y_1]$

This is the line representing all possible lotteries with expected *value* (not expected utility) of $pY_1 +(1-p)(Y_2)$.

The risk-averse individual will prefer the lottery which gives him $pY_1 +(1-p)(Y_2)$ *with certainty* to all other points on this "budget constraint."This lottery occurs where $Y_1 = Y_2 = pY_1 +(1-p)(Y_2)$. Draw a 45-degree line through the origin intersecting the "budget constraint." The intersection of those two lines will be the point of tangency of the person's indifference curve I_1 for this "budget constraint." We can call this point Q.

If point Q is preferred to all other points on the "budget constraint," then it follows that the indifference curve I_1 is *higher* than the "budget constraint" for all points other than Q. This means that I_1 has a convex-to-origin shape.

Thus, **risk aversion implies convex indifference curves,** and the statement is **true**.

Problem 11-4. Stephen has a VNM utility function $u(x) = x^{1/2}$, where x is his state-contingent wealth. His initial wealth is $160,000. He is considering buying fire insurance, because he faces a 0.05 chance of a small fire that would do $70,000 damage, and a 0.05 chance of a big fire that would do $120,000 damage. He can't suffer both types of fire, i.e. there's a 0.9 chance no fire occurs.

11-4a. What lottery does he face without insurance?

Solution 11-4a.

Stephen faces the following lottery: **{160000, 90000, 40000; 0.9, 0.05, 0.05}**

Without insurance, Stephen has the following expected wealth:

E(w) = 0.9*160000 + 0.05*90000 + 0.05*40000 = $150,500

Stephen has the following expected utility:

EU = 0.9*√(160000) + 0.05*√(90000) + 0.05*√(40000) =

$EU_{no\_insurance}$ = 385

11-4b. Suppose that fire insurance requires payment of a deductible of $7620 in the event of a fire. What lottery does Stephen face with insurance?

Solution 11-4b. The lottery Stephen faces with insurance (where P is the premium) is

{160000 - P, 160000 - P - 7620, 160000 - P - 7620; 0.9, 0.05, 0.05} =

{160000 - P, 152380 - P; 0.9, 0.1} That is, Stephen can expect to have 160000 - P in wealth with probability 0.9 and 152380 - P in wealth with probability 0.1.

11-4c. Suppose that fire insurance requires payment of a deductible of $7620 in the event of a fire. What is the most he will pay for full insurance?

Solution 11-4c. In the event of any fire, Stephen will have 160000 - 7620 = $152380 compensated to him by the insurance company - not counting the premium P. So his expected utility from this lottery will be

EU = 0.9*√(160000 - P) + 0.1√(152380 - P). This expected utility needs to be at least 385, the EU with no insurance. Thus, the maximum premium P* occurs where

385 = 0.9*√(160000 - P) + 0.1√(152380 - P)

Solving this equation by intersecting graphs on the TI-83 calculator,

we get P = $11004. Indeed, we have

0.9*√(160000 - 11004) + 0.1√(152380 - 11004) = 385

So the most Stephen will pay for full insurance is **$11004.**

Section 12

Principal-Agent Problems and Designing Contracts Under Asymmetric Information

Problem 12-1. Ludwig hires Frederic to work on a project that will yield $700 in revenue if it succeeds and $100 in revenue if it fails. Frederic's opportunity cost of working without substantial effort is $160. His additional cost of working hard is $50. If Frederic works without substantial effort, the probability that the project will succeed is 0.3. If he works hard, the probability that the project will succeed is 0.6. Design a contract whereby Ludwig will pay Frederic so that Frederic has an incentive to work hard.

Solution 12-1. Let x be the payment to Frederic in the event of the project's success.

Let y be the payment to Frederic in the event of the project's failure.

Let p = 0.6 be the probability of success if Frederic works hard.

Let q = 0.3 be the probability of success if Frederic does not work hard.

Let w = 160 be Frederic's base opportunity cost.

Let e = 50 be Frederic's additional cost of working hard.

Frederic will work hard if

(i): $(p-q)(x-y) \geq e$

(ii): $y + p(x-y) \geq w + e$

(i) means that the expected gain to the manager from the extra effort is at least as high as the cost to him of the extra effort he expends when working hard.

(ii) means that the expected earnings of the manager (y + p(x-y)) are greater than his opportunity cost w + e.

We set (p-q)(x-y) = e: (p-q)(x-y) = 0.3(x - y) = 50, so x - y = 166.66666667

We set y + p(x-y) = w + e. We substitute 166.66666667 for (x - y). Thus,

y + 0.6*166.66666667 = 160 + 50 and y = 110. Therefore, x = 110 + 166.66666667 = x = 276.6666666667.

So the optimal contract will be as follows.

Contract: Ludwig pays Frederic 276.6666666667 if the project succeeds and 110 if the project fails.

Problem 12-2. The conditions of this problem are given in Example 9.1 of Avinash K. Dixit's *Optimization in Economic Theory:* "An owner has to hire a manager to run a project. If a project succeeds, it will produce value V. The probability of success depends on the quality of the manager's work. Given high quality, the project will succeed with probability p, but low quality will reduce this to q. The basic salary needed to attract a manager is w. But he has to exert himself more to achieve high quality, and will do so only if he is paid a premium e. Both the owner and the manager are risk-neutral, that is, each maximizes the mathematical expectation of his mathematical returns (minus the money-equivalent cost of effort in the case of the manager)."

12-2a. If V = \$300, p = 50%, q = 41.67% (that's forty one and two-thirds) w \$120, and e = \$10, is it possible to design a contract that is optimal for both parties, under which the manager works hard? If so what is it, and if not why not?

Solution 12-2a.

Let x be the payment to the manager in the event of the project's success.

Let y be the payment to the manager in the event of the project's failure.

The manager will work hard if

(i): $(p-q)(x-y) \geq e$

(ii): $y + p(x-y) \geq w + e$

(i) means that the expected gain to the manager from the extra effort is at least as high as the cost to him of the extra effort he expends when working hard.

(ii) means that the expected earnings of the manager ($y + p(x-y)$) are greater than his opportunity cost $w + e$.

We set $(p-q)(x-y) = e$: $(p-q)(x-y) = 0.0833333333*(x-y) = 10$, so $x - y = 120$

We set $y + p(x-y) = w + e$. Thus, $y + 0.5*120 = 130$ and $y = 70$ and $x = 190$.

Thus, it is possible to design the following contract: Pay the manager 70 if the project fails, and pay him 190 if the project succeeds - if the manager agrees to work hard. This is the least that the manager would need to be paid in order to work hard.

12-2b. Suppose that the manager's efforts do not affect the probability of success, but the revenue generated if the project succeeds. If V(H) = \$300, V(L) = \$200, p = 50%, w = \$120 and e = $10, design an optimal contract under which the manager will work hard. Show that this is preferable for both parties.

Solution 12-2b.

If the project succeeds, the owner can observe ex post whether manager worked hard.

If the manager works hard, the owner's expected earnings from the project are pV(H) = 0.5*300 = 150

The owner would have to pay the manager at least the manager's opportunity cost w + e = 130, leaving the owner with 20 in profit.

If the manager does not work hard, the owner's expected earnings from the project are pV(H) = 0.5*200 = 100

The owner would have to pay the manager at least the manager's opportunity cost w = 120, leaving the owner with -20 in profit.

So the owner would prefer for the manager to work hard.

However, without any kind of special contract, the condition $(p-q)(x-y) \geq e$ does not hold for the manager, since here p = q, so (p - q) = 0, so (p-q)(x-y) = 0 and thus, $(p-q)(x-y) < e = 10$.

So the manager will not work hard without a contract, in which case the owner would simply have no reason to hire him - as if he did hire the manager, the owner could expect to lose money.

But the owner could get the manager to work hard via the following arrangement.

Contract: The owner pays the manager $130 unless the project yields a revenue of $200, in which case the owner pays the manager nothing.

For the manager, the expected gain if he works hard will be $130 - since he will get paid whether the project succeeds or fails in that case. His opportunity cost will be covered, so the manager will be willing to work.

If the manager does not work hard, however, his expected gain will be 0.5*130 + 0.5*0 = $65. The manager will get paid if the project fails but not if it succeeds. The expected gain of not working hard will not cover the manager's opportunity cost of $120, so the manager will not default on the contract. So the manager will work hard and get a guaranteed income of $130, while the owner can expect a profit of $20; both are in a superior position to the one they would have been in if there had been no contract.

Problem 12-3. Cuauhtemoc hires Tim to work on a project where the payoff to Cuauhtemoc if the project succeeds is $900 and the payoff if it fails is $800. If Tim works hard, the project has a 0.5 probability of succeeding. If Tim does not work hard, the project has a 0.45 probability of succeeding. Tim's base opportunity cost is $300, and his extra opportunity cost of working hard is $40. Can a contract be designed to mutual advantage so as to enable Tim to work hard? If so, what is that contract? If not, why not?

Solution 12-3. Let x be the payment to Tim in the event of the project's success.
Let y be the payment to Tim in the event of the project's failure.
Let p = 0.5 be the probability of success if Tim works hard.
Let q = 0.45 be the probability of success if Tim does not work hard.
Let w = 300 be Tim's base opportunity cost.
Let e = 40 be Tim's additional cost of working hard.

Tim will work hard if

(i): $(p-q)(x-y) \geq e$

(ii): $y + p(x-y) \geq w + e$

(i) means that the expected gain to the manager from the extra effort is at least as high as the cost to him of the extra effort he expends when working hard.

(ii) means that the expected earnings of the manager ($y + p(x-y)$) are greater than his opportunity cost w + e.

We set $(p-q)(x-y) = e$. Thus, $0.05(x-y) = 40$ and $(x - y) = \$800$
We set $y + p(x-y) = w + e$. Thus, $y + 0.5*800 = 340$ and $y = -\$60$, so $x = \$740$

So we propose the following contract:
Pay Tim $740 if the project succeeds and fine him $60 if the project fails.

However, this contract will not work because we have not considered the Cuauhtemoc's incentives. Cuauhtemoc will only want Tim to work hard if his expected profit from Tim working hard exceeds his expected profit from Tim not working hard.

That is, for the owner, it is necessary that

$$\mathbf{pV_{success} + (1-p)V_{failure} - w - e \geq qV_{success} + (1-q)V_{failure} - w}$$

In Cuauhtemoc's case, however,
$pV_{success} + (1-p)V_{failure} - w - e = 0.5*900 + 0.5*800 - 340 = 510$
$qV_{success} + (1-q)V_{failure} - w = 0.45*900 + 0.55*800 - 300 = 545$

So Cuauhtemoc can actually get a higher expected profit if Tim *does not work hard,* in which case a contract to get Tim to work hard would actually get Cuauhtemoc to lose money. Thus, **a mutually advantageous contract to get Tim to work hard cannot be designed.**

About Mr. Stolyarov

Gennady Stolyarov II (G. Stolyarov II) is an actuary, science-fiction novelist, independent philosophical essayist, poet, amateur mathematician, composer, and Editor-in-Chief of The Rational Argumentator, a magazine championing the principles of reason, rights, and progress.

In December 2013, Mr. Stolyarov published *Death is Wrong*, an ambitious children's book on life extension illustrated by his wife Wendy. *Death is Wrong* can be found on Amazon in paperback and Kindle formats.

Mr. Stolyarov has contributed articles to the Institute for Ethics and Emerging Technologies (IEET), The Wave Chronicle, Le Quebecois Libre, Brighter Brains Institute, Immortal Life, Enter Stage Right, Rebirth of Reason, The Liberal Institute, and the Ludwig von Mises Institute. Mr. Stolyarov also published his articles on Associated Content (subsequently the Yahoo! Contributor Network) from 2007 until its closure in 2014, in an effort to assist the spread of rational ideas. He held the highest Clout Level (10) possible on the Yahoo! Contributor Network and was one of its Page View Millionaires, with over 3.1 million views.

Mr. Stolyarov holds the professional insurance designations of Associate of the Society of Actuaries (ASA), Associate of the Casualty Actuarial Society (ACAS), Member of the American Academy of Actuaries (MAAA), Chartered Property Casualty Underwriter (CPCU), Associate in Reinsurance (ARe), Associate in Regulation and Compliance (ARC), Associate in Personal Insurance (API), Associate in Insurance Services (AIS), Accredited Insurance Examiner (AIE), and Associate in Insurance Accounting and Finance (AIAF).

Mr. Stolyarov has written a science fiction novel, *Eden against the Colossus*, a philosophical treatise, *A Rational Cosmology*, a play, *Implied Consent*, and a free self-help treatise, *The Best Self-Help is Free*. You can watch his YouTube Videos. Mr. Stolyarov can be contacted at gennadystolyarovii@gmail.com.